CS 188: Artificial Intelligence
Spring 2007

Lecture 6: CSP

2/1/2007

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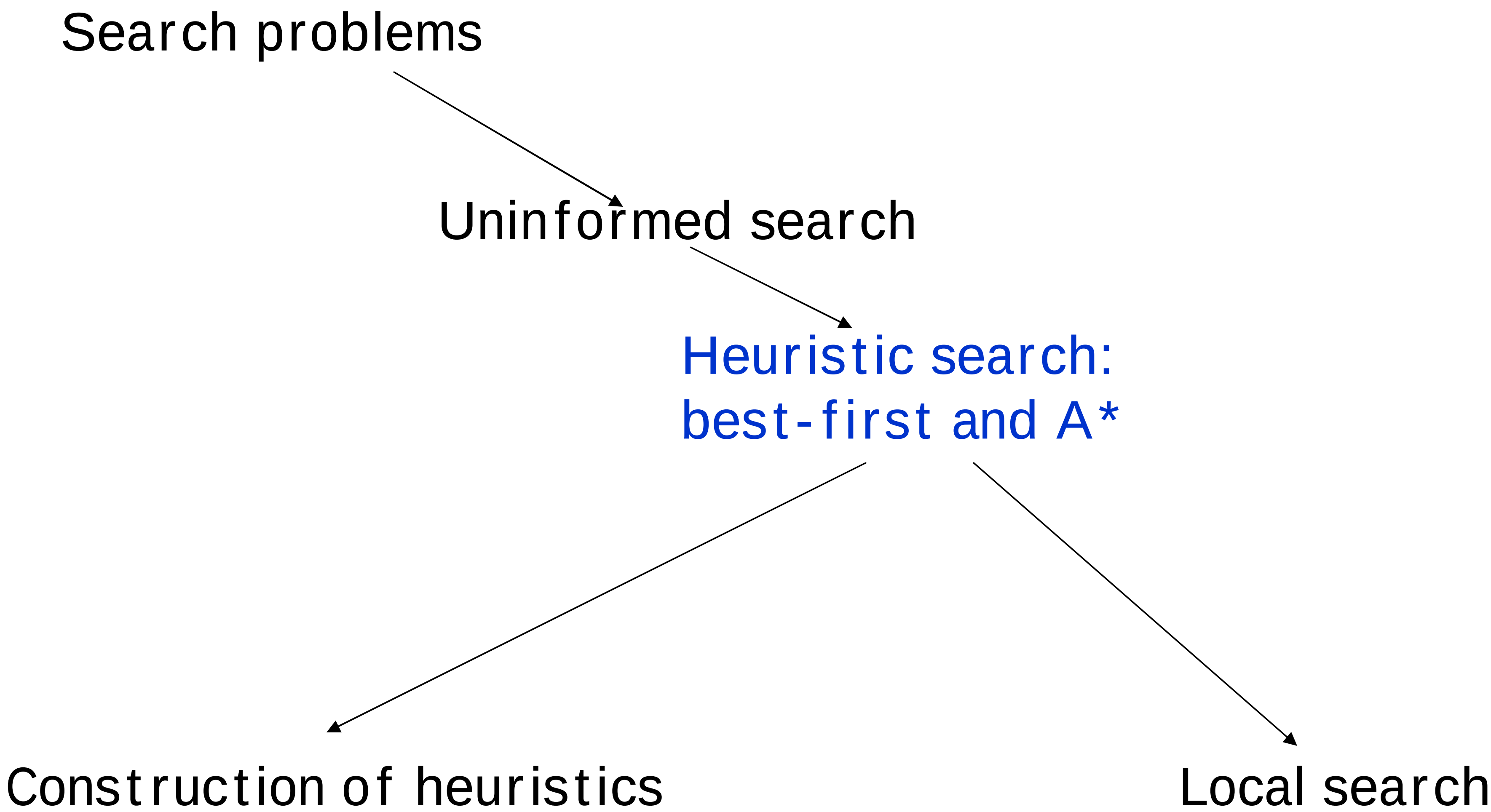
Many slides over the course adapted from Dan Klein, Stuart Russell or Andrew Moore

Announcements

Assignment 2 is up (due 2/12)

The past

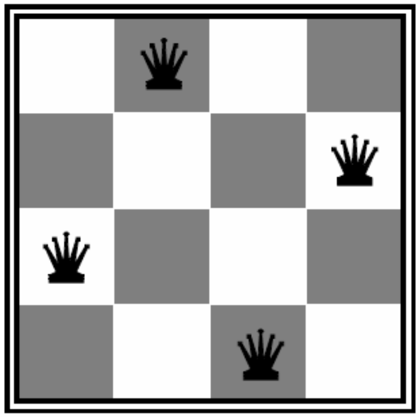

Search problems
Uninformed search
Heuristic search:
best-first and A*
Construction of heuristics
Local search

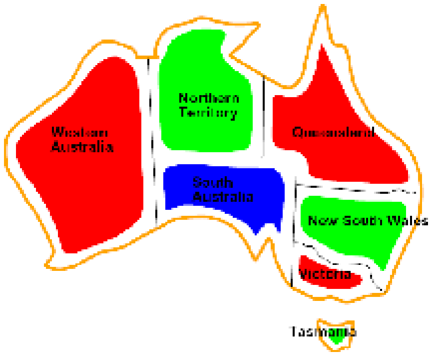
Today

- CSP
  - Formulation
  - Propagation
  - Applications

Constraint Satisfaction Problems

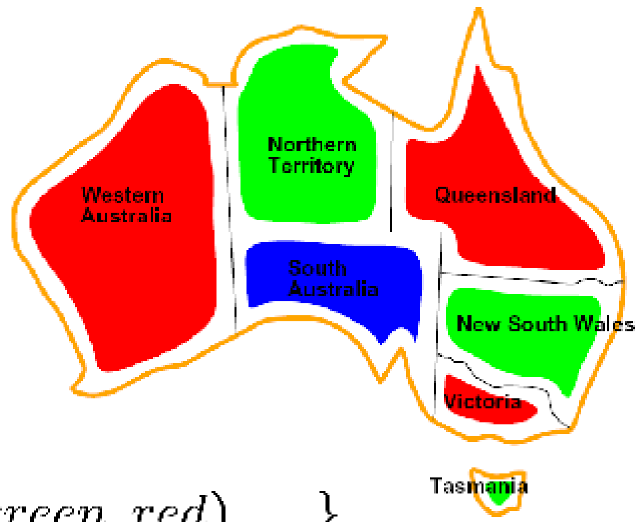
- Standard search problems:
  - State is a “black box”: any old data structure
  - Goal test: any function over states
  - Successors: any map from states to sets of states
- Constraint satisfaction problems (CSPs):
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms





Example: Map-Coloring

- Variables: $WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors



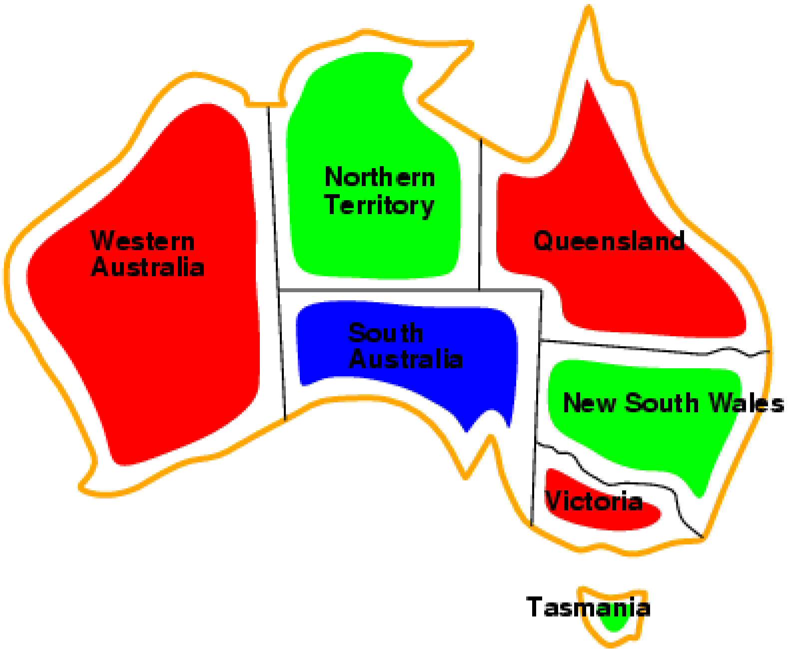
$$WA \neq NT$$

$$(WA, NT) \in \{(red, green), (red, blue), (green, red), \ldots\}$$

- Solutions are assignments satisfying all constraints, e.g.:

$$\{WA = red, NT = green, Q = red,$$
$$NSW = green, V = red, SA = blue, T = green\}$$

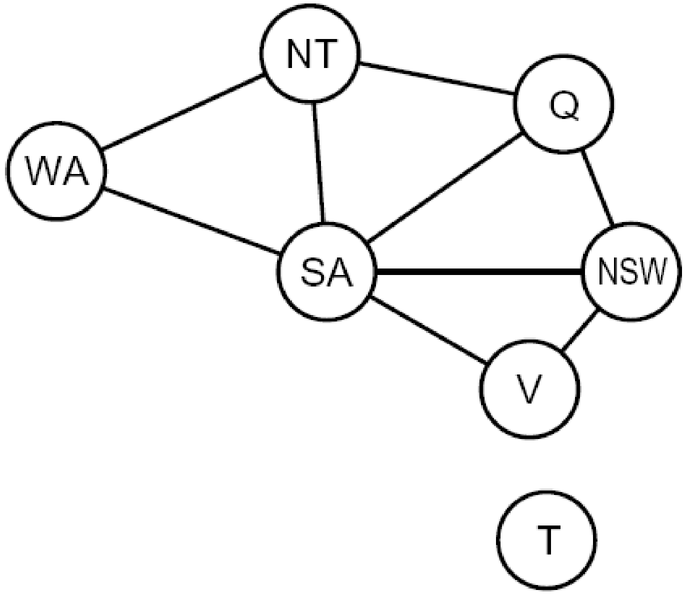
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



Example: Cryptarithmetic

- Variables:

  $$F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$$

- Domains:

  $$\{0,1,2,3,4,5,6,7,8,9\}$$

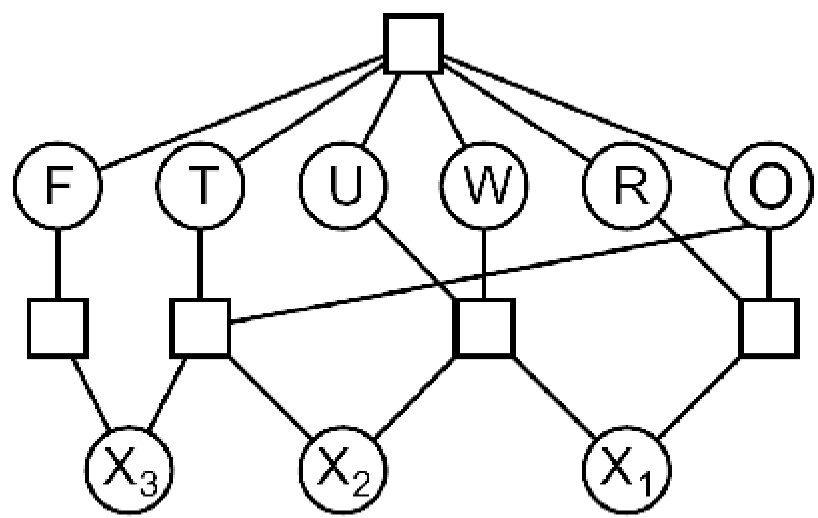
- Constraints:

  $$\text{alldiff}(F,T,U,W,R,O)$$

  $$O+O=R+10\cdot X_1$$

  $$\cdots$$

```
   T W O
+  T W O
--------
 F O U R
```



Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Need a *constraint language*, e.g., $StartJob_1 + 5 < StartJob_3$
    - Linear constraints solvable, nonlinear undecidable

- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods (see cs170 for a bit of this theory)

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains):

    $$SA \neq green$$

  - Binary constraints involve pairs of variables:

    $$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Floorplanning

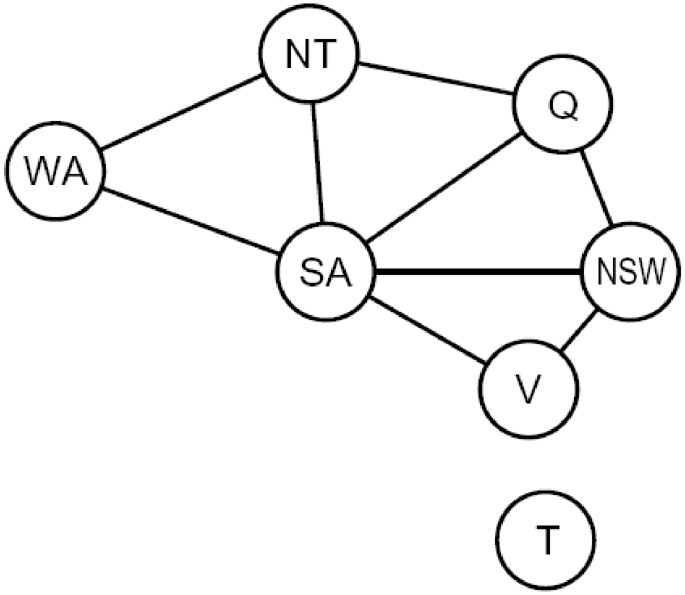
- Many real-world problems involve real-valued variables…

Standard Search Formulation

- Standard search formulation of CSPs (incremental)
- Let's start with the straightforward, dumb approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
    - fail if no legal assignment
  - Goal test: the current assignment is complete and satisfies all constraints

Search Methods

- What does DFS do?



- What's the obvious problem here?
- What's the slightly-less-obvious problem?

CSP formulation as search

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   à use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. b = (n - *l*)d at depth *l*, hence $n! \cdot d^n$ leaves

Backtracking Search

- Idea 1: Only consider a single variable at each point:
  - Variable assignments are commutative
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there?

- Idea 2: Only allow legal assignments at each point
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to figure out whether a value is ok

- Depth-first search for CSPs with these two improvements is called *backtracking search*

- Backtracking search is the basic uninformed algorithm for CSPs

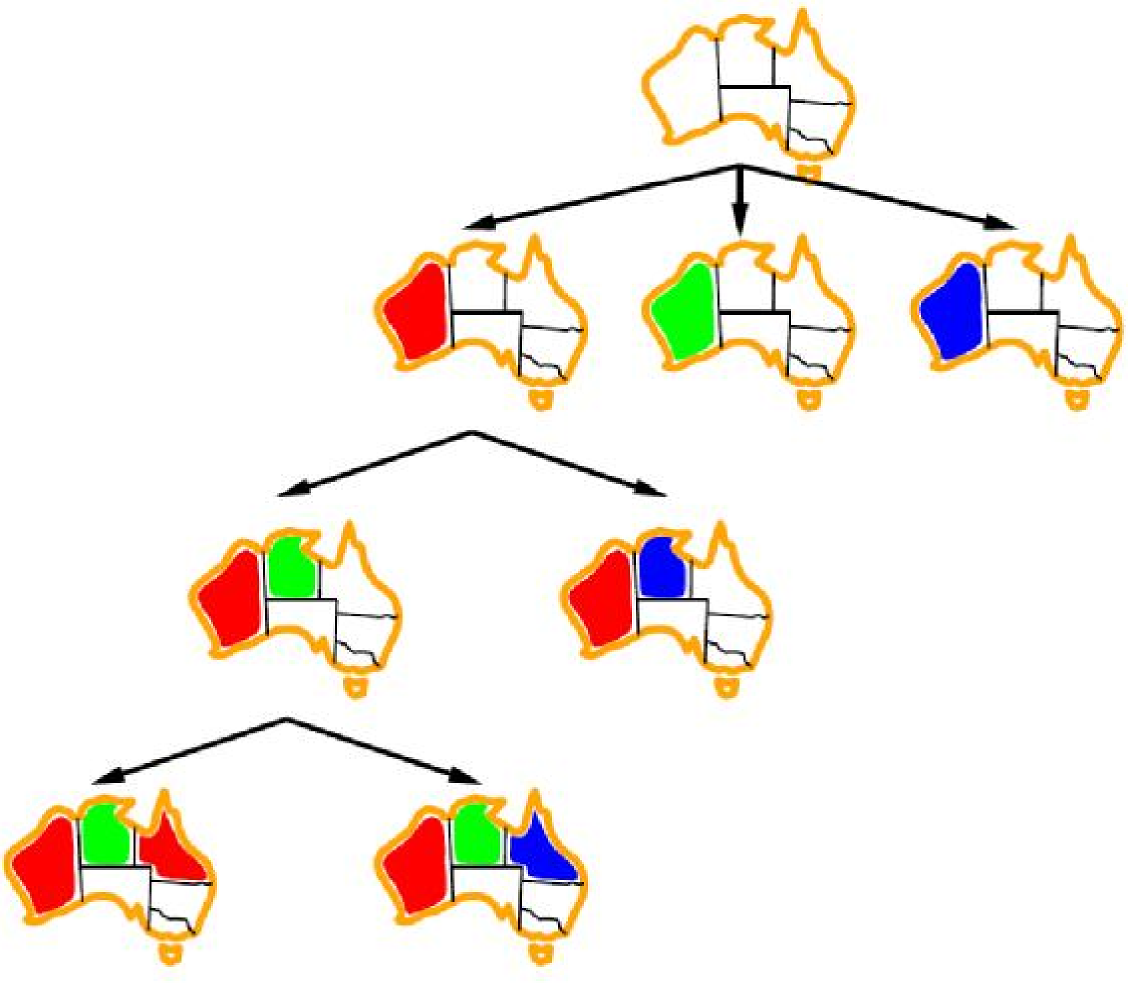
- Can solve n-queens for n $\approx$ 25

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
       if value is consistent with assignment given CONSTRAINTS[csp] then
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

- What are the choice points?

Backtracking Example

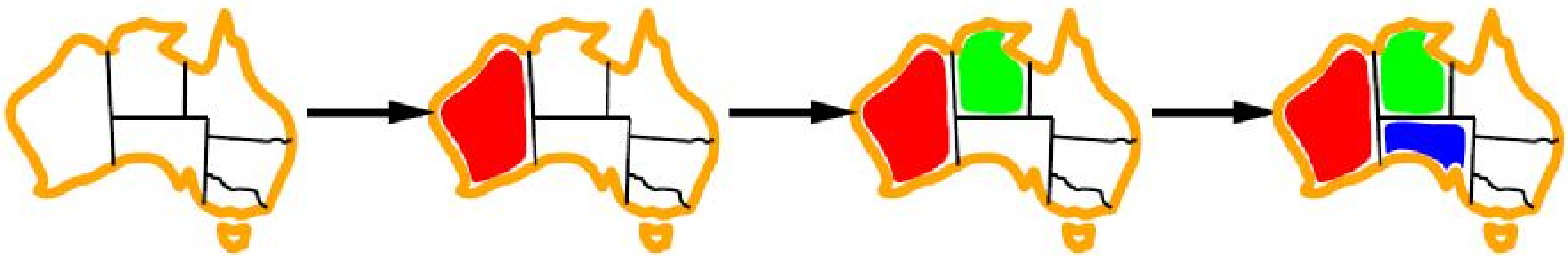


Improving Backtracking

- General-purpose ideas can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

Minimum Remaining Values

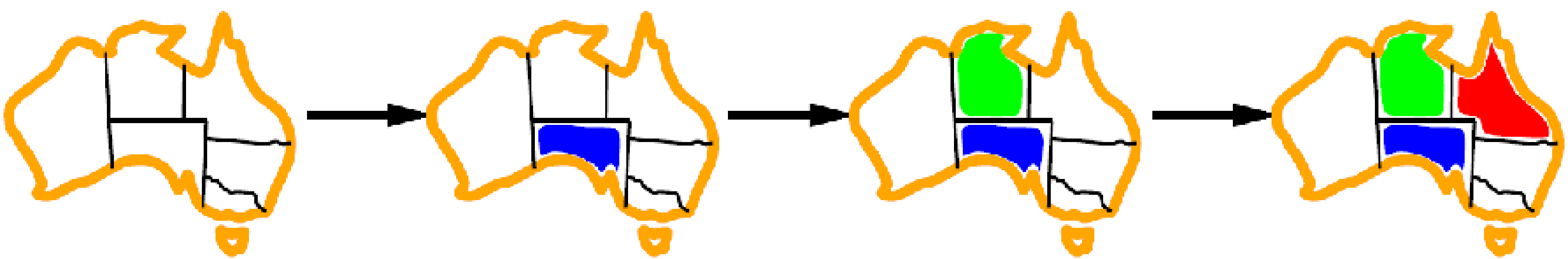
- Minimum remaining values (MRV):
  - Choose the variable with the fewest legal values



- Why min rather than max?
- Called most constrained variable
- "Fail-fast" ordering

Degree Heuristic

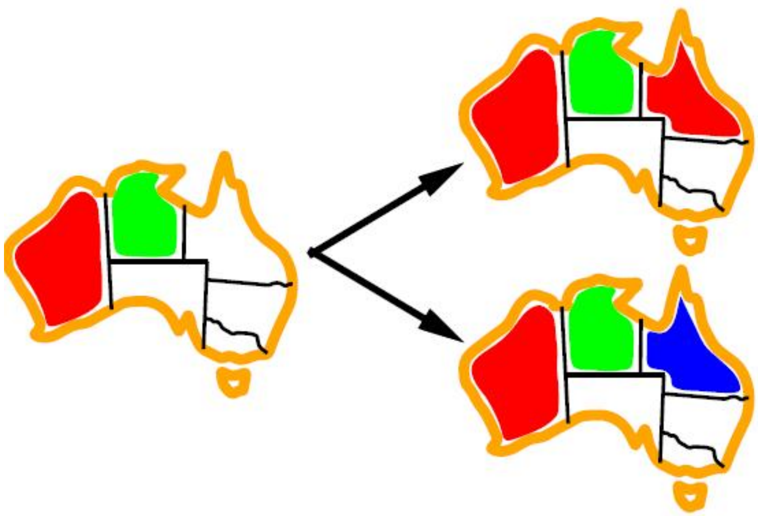
- Tie-breaker among MRV variables
- Degree heuristic:
  - Choose the variable with the most constraints on remaining variables



- Why most rather than fewest constraints?

Least Constraining Value

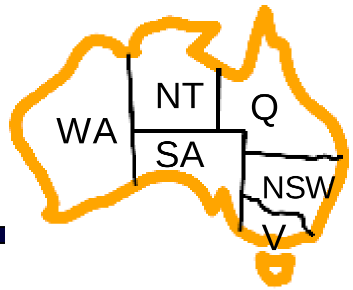
- Given a choice of variable:
  - Choose the *least constraining value*
  - The one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!



- Why least rather than most?

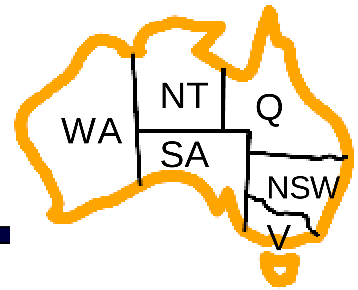
- Combining these heuristics makes 1000 queens feasible

Forward Checking

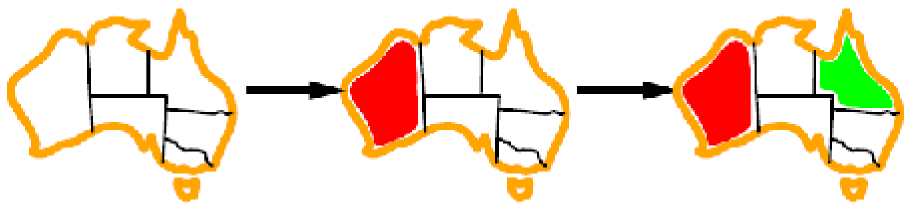


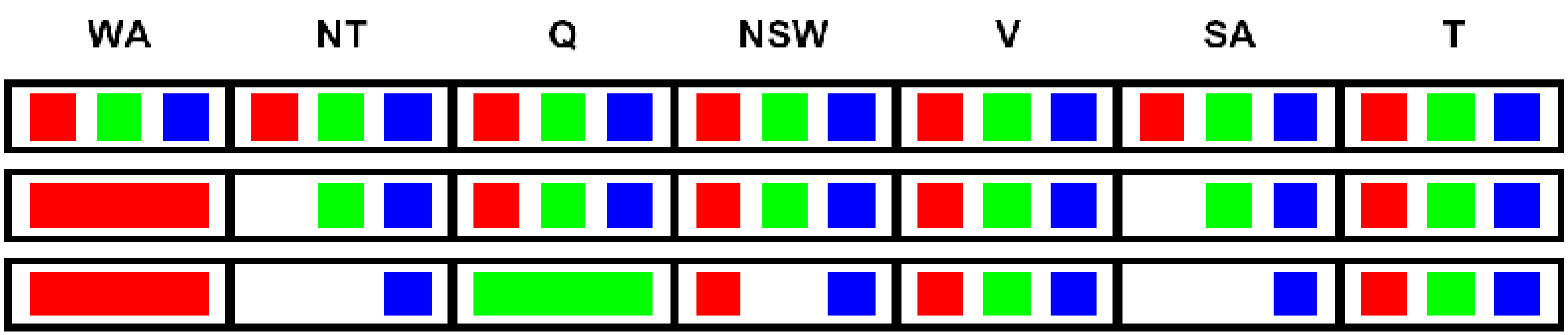
- Idea: Keep track of remaining legal values for unassigned variables
- Idea: Terminate when any variable has no legal values

Constraint Propagation



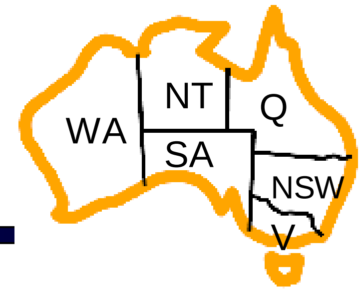
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



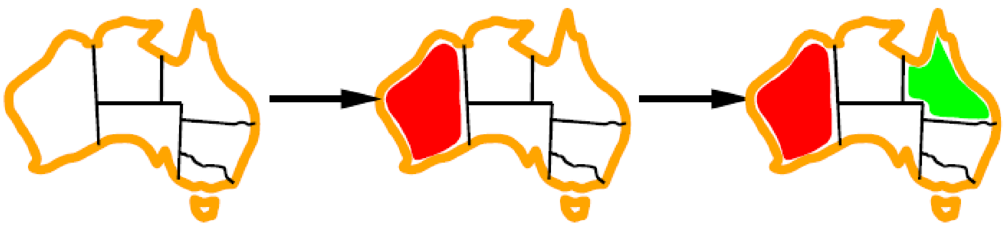


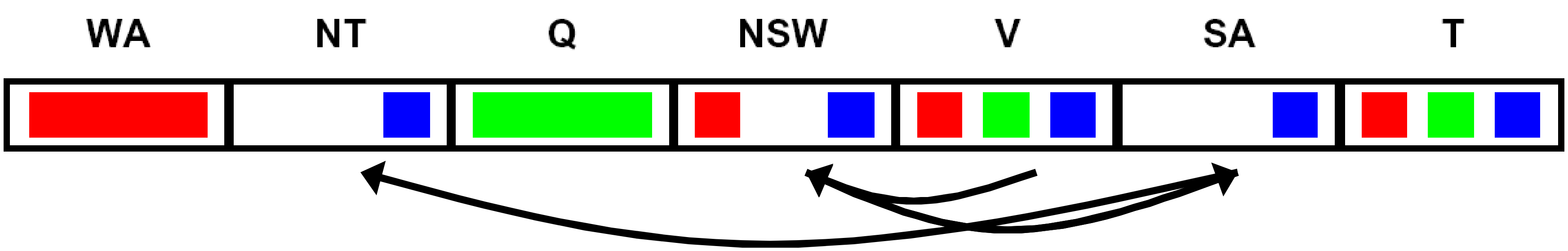
- NT and SA cannot both be blue!
- Why didn’t we detect this yet?
- *Constraint propagation* repeatedly enforces constraints (locally)

Arc Consistency



- Simplest form of propagation makes each arc *consistent*
  - X → Y is consistent iff for *every* value x there is *some* allowed y





- If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- What’s the downside of arc consistency?
- Can be run as a preprocessor or after each assignment

Arc Consistency

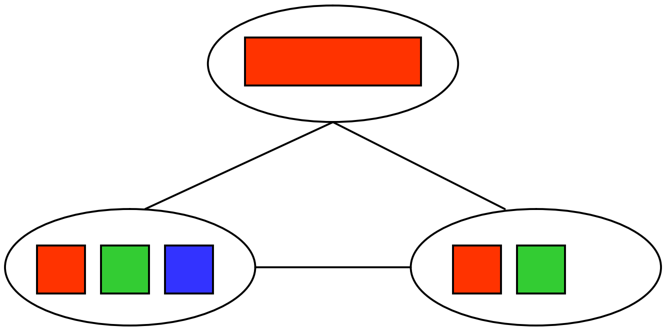
```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

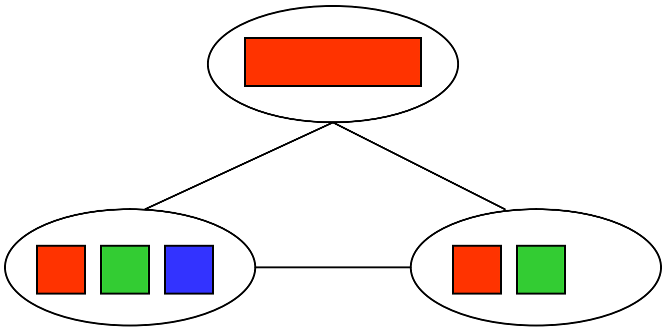
```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
  - N^2 arcs, each arc at most d times (till no values), checking is d^2
- … but detecting all possible future problems is NP-hard – why?

Summary: Consistency

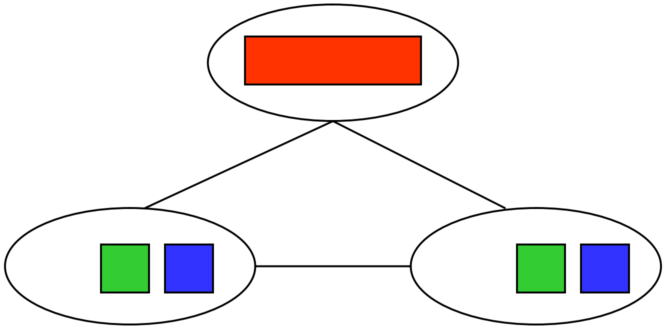
- Basic solution: DFS / backtracking
  - Add a new assignment
  - Check for violations
- Forward checking:
  - Pre-filter unassigned domains after every assignment
  - Only remove values which conflict with current assignments
- Arc consistency
  - We only defined it for binary CSPs
  - Check for impossible values on all pairs of variables, prune them
  - Run (or not) after each assignment before recursing
  - A pre-filter, not search!

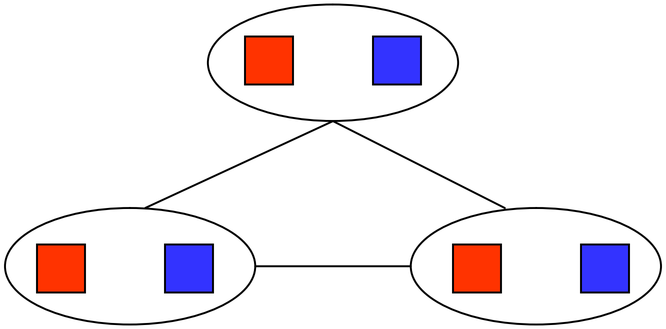




Limitations of Arc Consistency

- After running arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

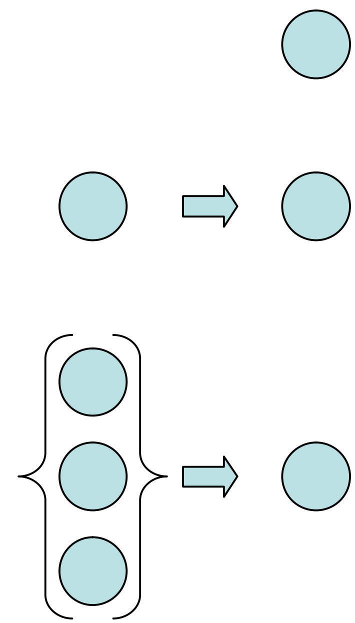




What went wrong here?

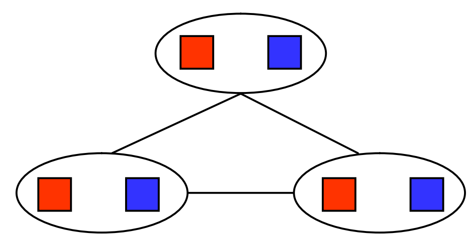
K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



- Higher k more expensive to compute

- (You need to know the k=2 algorithm)



Strong K-Consistency

- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. path consistency)

Iterative Algorithms for CSPs

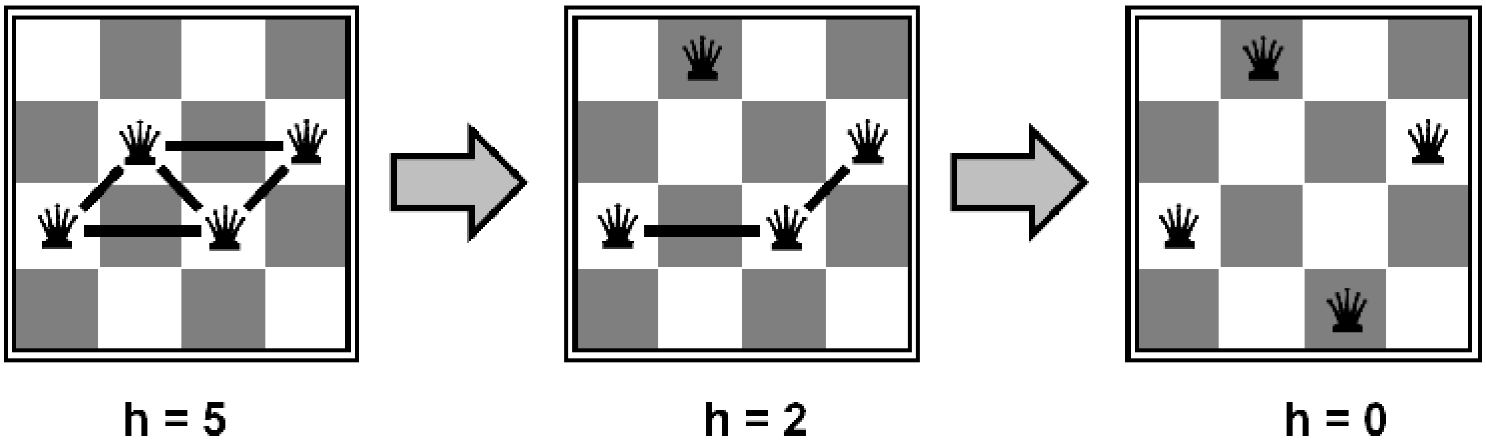
- Greedy and local methods typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - Allow states with unsatisfied constraints
  - Operators *reassign* variable values

- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:
  - Choose value that violates the fewest constraints
  - I.e., hill climb with h(n) = total number of violated constraints

Example: 4-Queens



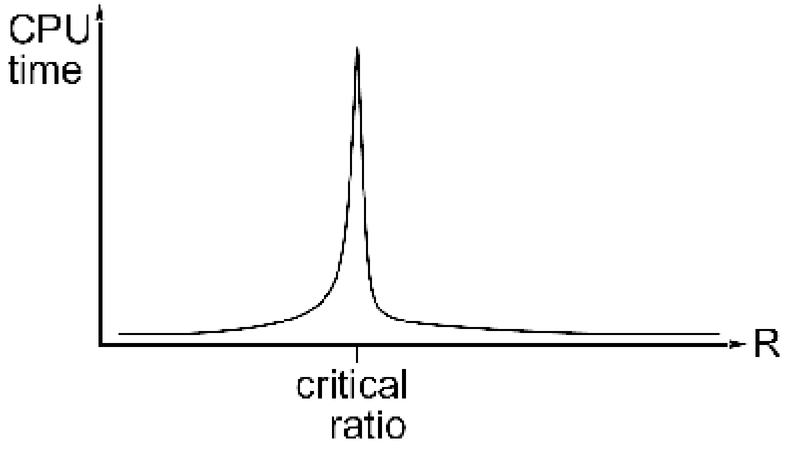
- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Example: Boolean Satisfiability

- Given a Boolean expression, is it satisfiable?
- Very basic problem in computer science

$$p_1 \wedge (p_2 \rightarrow p_3) \wedge ((\neg p_1 \wedge \neg p_3) \rightarrow \neg p_2) \wedge (p_1 \vee p_3)$$

- Turns out you can always express in 3-CNF

$$(p_1) \wedge (\neg p_2 \vee p_3) \wedge (p_1 \vee p_3 \vee \neg p_2) \wedge (p_1 \vee p_2 \vee p_3)$$

- 3-SAT: find a satisfying truth assignment

Example: 3-SAT

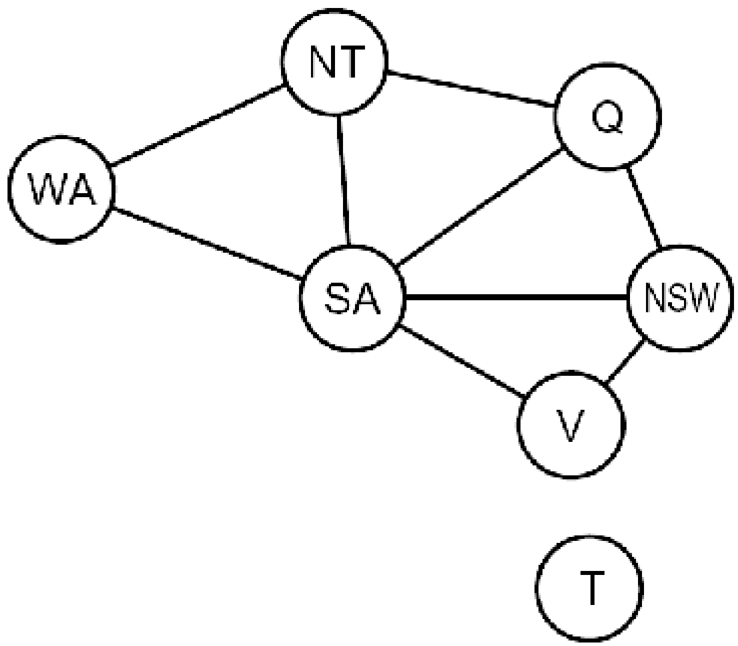
- Variables: $p_1, p_2, \cdots p_n$
- Domains: $\{\mathsf{true}, \mathsf{false}\}$
- Constraints:

$$p_i \vee p_j \vee p_k$$
$$\neg p_{i'} \vee p_{j'} \vee p_{k'}$$
$$\vdots$$
$$p_{i''} \vee \neg p_{j''} \vee \neg p_{k''}$$

Implicitly conjoined (all clauses must be satisfied)

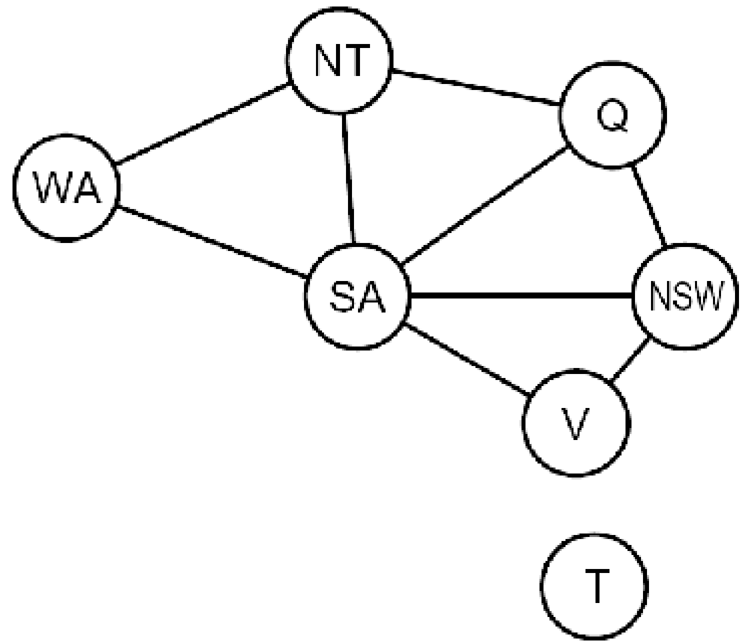
CSPs: Queries

- Types of queries:
  - Legal assignment
  - All assignments
  - Possible values of some query variable(s) given some evidence (partial assignments)

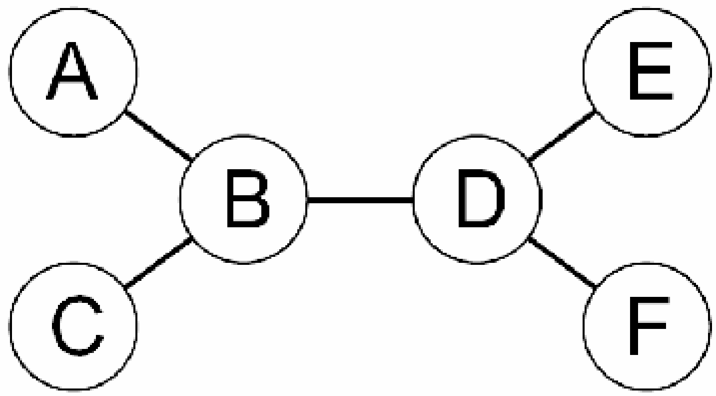


Problem Structure

- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph
- Suppose each subproblem has c variables out of n total
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec

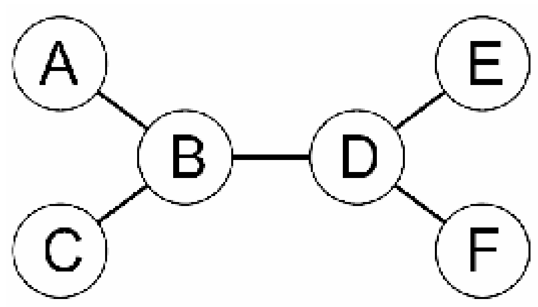


Tree-Structured CSPs

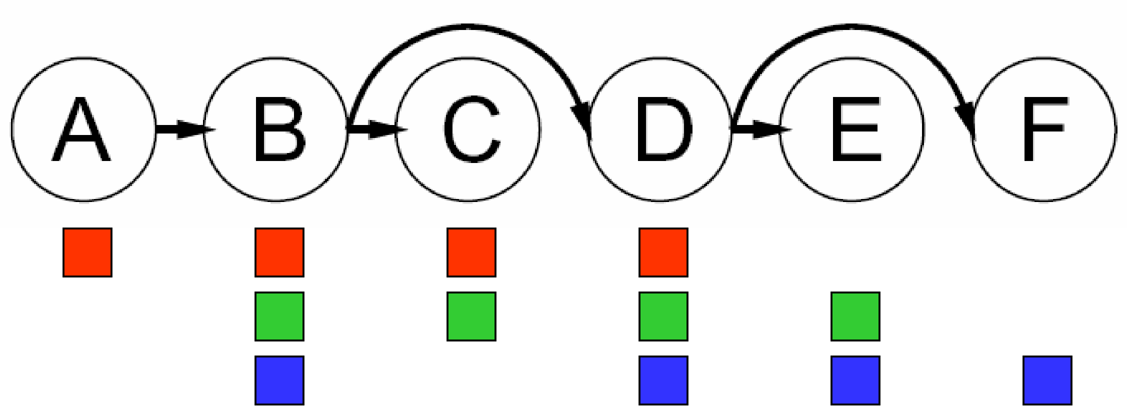


- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

Tree-Structured CSPs



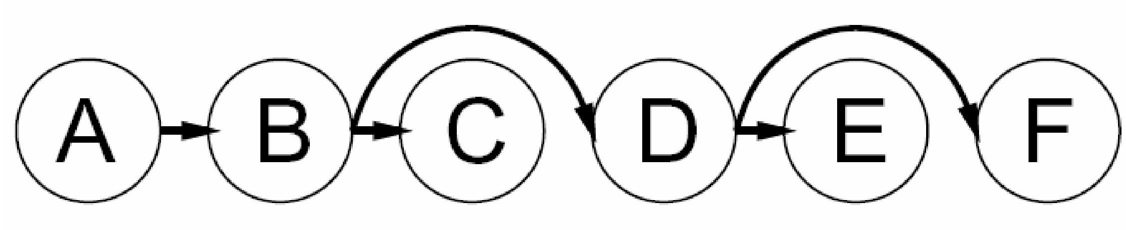
- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



- For $i = n : 2$, apply RemoveInconsistent(Parent(X_i),X_i)
- For $i = 1 : n$, assign X_i consistently with Parent(X_i)
- Runtime: $O(n\ d^2)$ (why?)

Tree-Structured CSPs

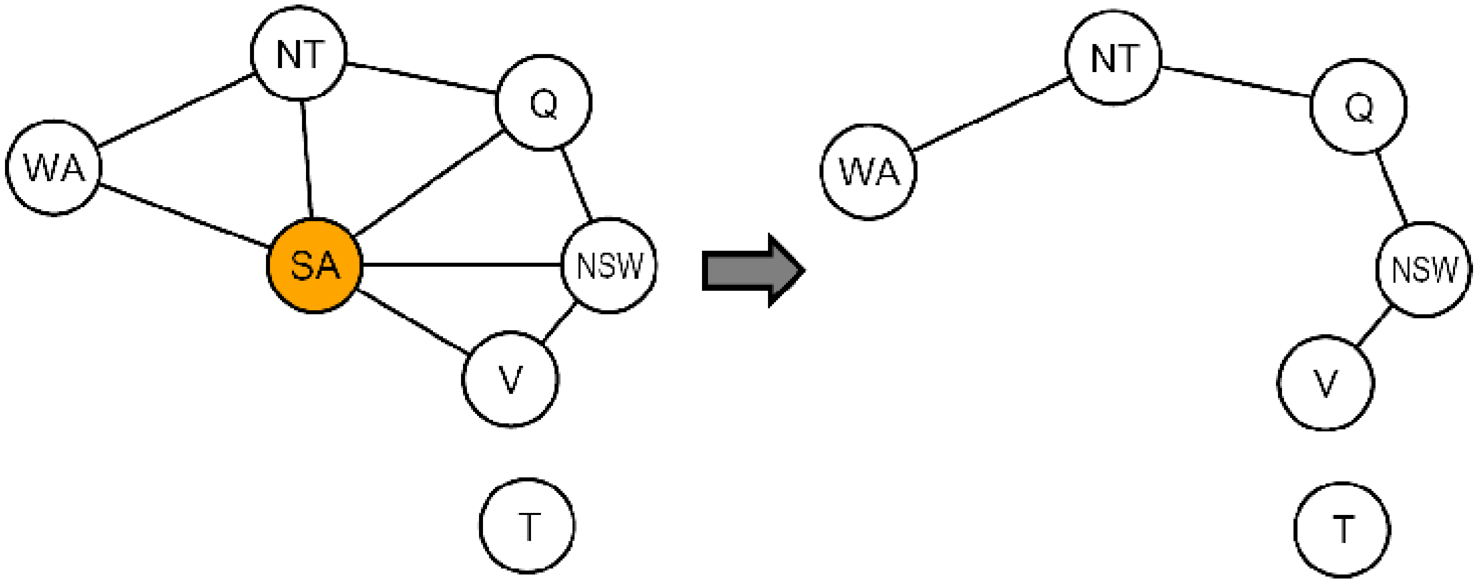
- Why does this work?
- Claim: After each node is processed leftward, all nodes to the right can be assigned in any way consistent with their parent.
- Proof: Induction on position



- Why doesn't this algorithm work with loops?

- Note: we'll see this basic idea again with Bayes' nets and call it belief propagation

Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c

CSP Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = depth-first search with one legal variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The constraint graph representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice